A Naming System for Hyperoperator-related Functions up to $f_{\varepsilon_0}(n)$

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1 Section 1: Introduction (A Game With an Issue)

Hyperoperators are the common name given to the continuation of mathematical operators past the standard addition, multiplication and exponentiation. The most well known of these is tetration, although the list continues with pentation, hexation, and so on. Each operator a in this sequence roughly corresponds to $n\{a\}m$ where $\{a\}$ refers to a up arrows, or approximately $f_a(n)$ in the fast-growing hierarchy, or $\{n, m, a\}$ in array notation. The natural next step, if you are intent on surpassing the limit of the hyperoperator list, is to recurse on it. Doing so brings us to something like $n\{n\}n$, where the hyperoperator used increases as n increases. This function grows at an equivalent rate to Ackermann's fast-growing function, a common example of a function that is computable but not primitive recursive. Such a function is equivalent to {n,n,n} in array notation, and roughly equivalent to $f_\omega(n)$ in FGH.

In 2021 I created Endless Stairwell, an incremental game designed mainly to utilise the full scale of the JS library ExpantaNum.js and hold the largest numbers ever used in a major game. The numbers of endless Stairwell cap at around $\{10, 10^{15}, 1, 2\}$, roughly equivalent to $f_{\omega+1}(10^{15})$ in FGH. I added this game to an existing online list of “incremental game limits”, at which point I didn’t have a name for the growth rate of the game’s largest numbers. No problem, the googology wiki helpfully lists expansion as the function name for $\{a, b, 1, 2\} \approx f_{\omega+1}(b)$. This name was defined by Jonathan Bowers in his page on the Exploding Array Function some time around 2008. Unfortunately however, no such name exists for $f_\omega(n)$. In this page I seek to develop and propose a continuous system of names to fill this gap and continue further.

2 Section 2: The Basic System

Since the first recursion on the existing named hyperoperators is at $f_\omega(n)$, it’d be fair to call it **megation** (from ”omega”). I am aware that this term is already in use to refer to some other specific things, but they appear to be niche and informal.

An exact definition for megation is as follows: a megated to $b = \{a, a, b\}$ in array notation. This gives it the appropriate growth rate of $f_\omega(n)$. The prefix version of megation is mega-, which will be useful for larger growth rates. The next growth rate is expansion, or $\{a, b, 1, 2\} \approx f_{\omega+1}(n)$. The name for this in my system is mega-addition. After this comes mega-multiplication ($\{a, b, 2, 2\} \approx f_{\omega+2}(n)$), then mega-exponentiation ($\{a, b, 3, 2\} \approx f_{\omega+3}(n)$), and so on. This too can be recursed on with **duomegation** (megation with the prefix duo-), which is equal to $\{a, a, b, 2\} \approx f_{\omega 2}(n)$. Bowers defines $\{a, b, 1, 3\}$ as ”explosion,” although in my system it is called duomega-addition. From here on we can make continuing the system easier to understand by breaking down the related ordinals in FGH.

Every ordinal below ε_0 can be thought of as a series of decreasing segments $x_1 + x_2 + x_3 + \dots$ where each segment x is either a more simple ordinal, with the exception of the last segment, which can sometimes be a finite number[1]. An example of this is $\omega^2 + \omega 2 + 1$, with the segments being ω^2, $\omega 2$, and 1. These segments, when simplified, can be thought of as being composed of a tetrational factor, an exponential factor, and a multiplicative factor. For example, $(\omega^3)2$ can be said to have an exponential factor of 2 and a multiplicative factor of 1 (and a tetrational factor of 0) (Side note: this means that each segment could be displayed as an array for each factor, and thus every ordinal below ε_0 can be displayed using a 2-dimensional array[proof needed], which would be an unusual notation that may be worthy of development at some point). Each segment in the ordinal can be thought of as equivalent to a segment in my naming system. To take an existing example: Mega-addition ($f_{\omega+1}(n)$) has two segments, the mega (ω) segment and the addition (1) segment, separated by a hyphen. It can be said for simplicity that mega-addition has the ”equivalent ordinal” $\omega + 1$.

Functions with equivalent ordinals of multiples of ω use the standard prefixes for numerals (duo-, tri-, quadri-, quinti-, sexti-, septi-, octi-, noni-, deci-, etc.).

[1]Upon further consideration I believe that this system does not account for all ordinals below ε_0, however since it applies to most major ones this segment can remain for ease of understanding.

For example, trimegation ($\{a, a, b, 3\}$) has the equivalent ordinal $\omega 3$ and is thus roughly equivalent to $f_{\omega 3}(n)$.

Here is a table for some names based on the system as defined so far:

Function name	**Array equivalent**	**FGH growth rate (approximate)**
Addition		$f_1(n)$
Multiplication		$f_2(n)$
Exponentiation	$\{a, b\}$	$f_3(n)$
Tetration	$\{a, b, 2\}$	$f_4(n)$
Megation	$\{a, a, b\}$	$f_\omega(n)$
Mega-addition	$\{a, b, 1, 2\}$	$f_{\omega+1}(n)$
Mega-multiplication	$\{a, b, 2, 2\}$	$f_{\omega+2}(n)$
Duomegation	$\{a, a, b, 2\}$	$f_{\omega 2}(n)$
Duomega-addition	$\{a, b, 1, 3\}$	$f_{\omega 2+1}(n)$
Trimegation	$\{a, a, b, 3\}$	$f_{\omega 3}(n)$
Quadrimegation	$\{a, a, b, 4\}$	$f_{\omega 4}(n)$

The limit for this is $\{a, a, a, b\} \approx f_{\omega^2}(n)$. It is obvious where to go from here.

3 Continuation up to $f_{\varepsilon_0}(n)$

The next step after multiples of ω is **expomegation** (expo- from exponentiation), which is defined as $\{a, a, a, b\}$ in array notation. This means it has a growth rate of $f_{\omega^2}(n)$. Something important to note is that "duoexpomegation" is not $f_{(\omega^2)2}(n)$, but instead $f_{\omega^3}(n)$, and expoduomegation is $f_{(\omega^2)2}(n)$. This is where the comparison to segments starts to become more clearly useful; $f_{\omega^2+\omega+1}(n)$ would be expomega-mega-addition.

The final step towards reaching $f_{\varepsilon_0}(n)$ is to give names for tetrational (power-tower) ordinals such as ω^ω and ω^{ω^ω}. Although ordinal tetration is not well-defined, ordinals such as ω^ω can be informally simplified into $\omega \uparrow\uparrow 2$ (or ${}^2\omega$), making it more clear that the ordinal has a tetrational factor. To refer back to the factors with another example: $\omega^{(\omega^3)4}$ has a tetrational factor of 1, an exponential factor of 2, and a multiplicative factor of 3. These factors make it easy to create function names; **tetramegation** (tetra- from tetration) is defined as $\{a, b[2]2\}$ in array notation (specifically Bird's array notation). It has a growth rate of $f_{\omega^\omega}(n)$. Duotetramegation has a corresponding ordinal of ω^{ω^ω} and is defined as $\{a, b[1, 2]2\}$.

This system ends at $f_{\varepsilon_0}(n)$, the function for which I am going to give two different names: The special name **Epsilonation** (See Section 4), and the extrapolated name **Pentamegation** (See Section 5).

4 Special Names for Functions Past $f_{\varepsilon_0}(n)$

This section is a bit of a quick bonus, since these function names don't follow any sort of continuous system.

Epsilonation is defined as $\{a, b[[1]]2\} = \{a, b[1\backslash 2]2\}$. It has the equivalent ordinal of ε_0 and has a growth rate of $f_{\varepsilon_0}(n)$.

Zetation is defined as $\{a, b[1\backslash 1\backslash 2]2\}$. It has the equivalent ordinal of ζ_0 and has a growth rate of $f_{\zeta_0}(n)$.

Etation is defined as $\{a, b[1\backslash 1\backslash 1\backslash 2]2\}$. It has the equivalent ordinal of η_0 and has a growth rate of $f_{\eta_0}(n)$.

Gammation is defined as $\{a, b[1/2]2\}$. It has the equivalent ordinal of Γ_0 and a growth rate of $f_{\Gamma_0}(n)$, or $f_{\varphi(1,0,0)}(n)$ using the extended Veblen function.

5 Extrapolated Names for Functions Past $f_{\varepsilon_0}(n)$

To create a continuous naming system beyond the limit of defined ordinal arithmetic we can use a theoretical ill-defined system that equates further ordinals with higher hyperoperators. If we do so, it is easy to continue the pattern of expomegation and tetramegation with **pentamegation**, which would be defined as $\{a, b[[1]]2\} = \{a, b[1\backslash 2]2\}$ and have a growth rate of $f_{\varepsilon_0}(n)$. You could say that ε_0 has a pentational factor if you wish. Following this, duopentamegation would be defined as $\{a, b[1\backslash 1[1\backslash 2]2]2\}$ and have a growth rate of $f_{\varepsilon_{\varepsilon_0}}(n)$. Hexamegation would be defined as $\{a, b[1\backslash 1\backslash 2]2\}$ and have a growth rate of $f_{\zeta_0}(n)$.

We appear to be running into the exact issue of recursion that warranted this whole naming scheme. No problem, we can just plug our naming scheme into itself, giving us **megamegation**. Megamegation is defined as $\{a, b[1[2]\backslash 2]2\} = \{a, b[1\backslash\backslash 2]2\}$ and has a growth rate of $f_{\varphi(\omega,0)}(n)$ using the Veblen function. To continue this pattern, it would be wise to "simplify" our ordinals into a more understandable (yet ill-defined) pattern. If we replace ε_0 with $\omega \uparrow\uparrow \omega = \omega \uparrow\uparrow\uparrow 2$ and replace ζ_0 with $\omega \uparrow\uparrow\uparrow \omega = \omega\{4\}2$, we could say that $\varphi(\omega, 0) = \omega\{\omega\}\omega = \{\omega, \omega, \omega\}$. In fact, if we define $\varphi(\omega, 0)$ as being equal to $f_\omega(\omega)$, we can plug FGH into itself and create $f_{f_\omega(\omega)}(n)$ as a substitute for $f_{\varphi(\omega,0)}(n)$. Doing so makes calling our function name "megamegation" more clear. And now that it's clarified, we can continue further.

Duomegamegation has a growth rate of $f_{f_{\omega 2}(\omega)}(n)$, which I believe would be equal to $f_{\varphi(\omega 2,0)}(n)$. If so, it would be defined as $\{a, b[1\backslash\backslash 1\backslash\backslash 2]2\}$.

Expomegamegation has a growth rate of $f_{f_{\omega^2}(\omega)}(n)$, which I believe would be equal to $f_{\varphi(\omega^2,0)}(n)$. If so, it would be defined as $\{a, b[1[3]\backslash 2]2\}$.

Tetramegamegation has a growth rate of $f_{f_{\omega^\omega}(\omega)}(n)$, which I believe would be equal to $f_{\varphi(\omega^\omega,0)}(n)$. If so, it would be defined as $\{a, b[1[1, 2]\backslash 2]2\}$.

And finally, **megamegamegation** has a growth rate of $f_{f_{\varphi(\omega,0)}(\omega)}(n)$, which I believe would be equal to $f_{\varphi(\varphi(\omega,0),0)}(n)$. If so, it would be defined as $\{a, b[1[1\backslash\backslash 2]\backslash 2]2\}$.